ON ORTHOGONALITY, RATIONAL APPROXIMATION, QUADRATURE AND EXPONENTIAL ANALYSIS, IN ONE AND MORE VARIABLES

ANNIE CUYT

Abstract

We establish connections between the concept of orthogonal polynomials and several numerical techniques from rational approximation, Gaussian quadrature and exponential analysis, both in one as well as in several variables.

1. Univariate case

For a linear functional $L : \mathbb{C}[t] \to \mathbb{C} : t^i \to e_i$, a sequence of orthogonal polynomials $\{V_m(z)\}_{m\in\mathbb{N}}$ can be defined by

$$L\left(t^i V_m(t)\right) = 0, \qquad i = 0, \ldots, m-1.$$

In [7] these formally orthogonal polynomials are called Hadamard polynomials.

With the $V_m(z)$ we can define associated polynomials

$$W_{m-1}(z) = L\left(\frac{V_m(z) - V_m(t)}{z - t}\right)$$

and reverse polynomials

$$\tilde{V}_m(z) = z^m V_m(1/z).$$

The Padé approximant $[m-1/m]_F$ of degree $m-1$ in the numerator and degree m in the denominator to the formal power series

$$F(z) = \sum_{i=0}^{\infty} e_i z^i$$

precisely equals $\tilde{W}_{m-1}(z)/\tilde{V}_m(z)$. Hence the denominator of this Padé approximant is closely related to the orthogonal polynomial $V_m(z)$.

When the e_i are so-called moments, for instance

$$e_i = \int_{-1}^{1} w(t) t^i \, dt, \qquad 0 < \int_{-1}^{1} w(t)\, dt,$$

then

$$F(z) = \int_{-1}^{1} w(t) \frac{1}{1-tz}\, dt$$

and the m-point Gaussian quadrature rule

$$\int_{-1}^{1} w(t) \frac{1}{1-tz}\, dt \approx \sum_{i=1}^{m} A_i^{(m)} \frac{1}{1 - z_i^{(m)} z},$$

approximating $F(z)$, equals the $[m-1/m]_F$ Padé approximant. The Gaussian nodes $z_i^{(m)}$ are the zeroes of $V_m(z)$ and the weights $A_i^{(m)}$ are given by

$$A_i^{(m)} = \frac{W_{m-1}(z_i^{(m)})}{V_m'(z_i^{(m)})}, \qquad i = 1, \ldots, m.$$

Since this Gaussian quadrature rule exactly integrates polynomials of degree $2m-1$, we also have

$$e_j = \sum_{i=1}^{m} A_i^{(m)} \left(z_i^{(m)}\right)^j, \qquad j = 0, \ldots, 2m-1.$$

Hence the nodes and weights can be obtained as the solution of the exponential analysis or Prony problem [6]

$$e_j = \sum_{i=1}^{m} A_i^{(m)} \exp(j\phi_i^{(m)}), \qquad z_i^{(m)} = \exp(\phi_i^{(m)}), \quad j = 0, \ldots, 2m-1,$$

where only m and the e_j are given. The $z_i^{(m)}$ are the generalized eigenvalues of a Hankel structured generalized eigenvalue problem and the $A_i^{(m)}$ are the solution of a Vandermonde structured linear system [8].

2. Multivariate case

The concept of the formally orthogonal polynomial $V_m(z)$ is generalized in [4], for different radial weight functions, to so-called spherical orthogonal polynomials. The latter differ from several other definitions of multivariate orthogonal polynomials, in that they preserve the connections laid out above.

Homogeneous multivariate Padé approximants, as defined in [2, 3], can also be obtained from the spherical orthogonal polynomials in a similar way as described here. The homogeneous definition satisfies a very strong projection property, in the sense that this multivariate Padé approximant reduces to the univariate Padé approximant on every one-dimensional subspace.

A whole lot of Gaussian cubature rules on the disk can be united in a single approach [1] when developing the existing rules from these spherical orthogonal polynomials. What's more, the nodes and weights of such Gaussian cubature rules on the disk can be obtained as the solution of a multivariate Prony-like system of interpolation conditions [1]. And this brings us to the next connection.

Prony's result that an m-term exponential analysis problem can be solved uniquely from $2m$ samples e_i, is a one-dimensional result. In [5] this result is proven, more than 200 years later, to hold for higher dimensions $d > 1$: a multivariate linear combination of m exponential terms with unknown inner product exponents can, under mild conditions, be fitted using only $(d+1)m$ data.

Keywords: Orthogonal polynomials, Rational approximation, Quadrature rules, Exponential analysis
AMS Classification: 42C05, 41A21, 65D32, 65T40.

Bibliography

[1] B. Benouahmane, A. Cuyt, and I. Yaman. Near-minimal cubature formulae on the disk. *IMA J. Numer. Anal.*, 39(1):297–314, 2019. (Published online December 08, 2017).

[2] A. Cuyt. Multivariate Padé approximants. *J. Math. Anal. Appl.*, 96:283–293, 1983.

[3] A. Cuyt. Multivariate Padé approximants revisited. *BIT*, 26:71–79, 1986.

[4] A. Cuyt, B. Benouahmane, and B. Verdonk. Spherical orthogonal polynomials and symbolic-numeric Gaussian cubature formulas. In M. Bubak et al., editors, *LNCS 3037*, pages 557–560, Berlin, 2004. Springer-Verlag.

[5] Annie Cuyt and Wen-shin Lee. Multivariate exponential analysis from the minimal number of samples. *Adv. Comput. Math.*, 44:987–1002, 2018. (Published online November 16, 2017. Toolbox and experiments downloadable.).

[6] R. de Prony. Essai expérimental et analytique sur les lois de la dilatabilité des fluides élastiques et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures. *J. Ec. Poly.*, 1(22):24–76, 1795.

[7] P. Henrici. *Applied and computational complex analysis I.* John Wiley & Sons, New York, 1974.

[8] L. Weiss and R.N. McDonough. Prony's method, Z-transforms, and Padé approximation. *SIAM Rev.*, 5:145–149, 1963.

Annie Cuyt,
Dept Computer Science, Universiteit Antwerpen (BE), Middelheimlaan 1, 2020-Antwerpen, Belgium.
Division of Computing Science, University of Stirling (UK).
Dept Electrical Engineering, Technische Universiteit Eindhoven (NL).
annie.cuyt@uantwerpen.be